

# Pair creation by a photon in an electric field

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## Abstract

The process of pair creation by a photon in a constant and homogeneous electric field is investigated basing on the polarization operator in the field. The total probability of the process is found in a relatively simple form. At high energy the quasiclassical approximation is valid. The corrections to the standard quasiclassical approximation (SQA) are calculated. In the region relatively low photon energies, where SQA is unapplicable, the new approximation is used. It is shown that in this energy interval the probability of pair creation by a photon in electric field exceeds essentially the corresponding probability in a magnetic field. This approach is valid at the photon energy much larger than "vacuum" energy in electric field:  $\omega \gg eE/m$ . For smaller photon energies the low energy approximation is developed. At  $\omega \ll eE/m$  the found probability describes the absorption of soft photon by the particles created by an electric field.

# 1 Introduction

Pair creation by a photon in an electromagnetic field is the basic QED reaction which can play the significant role in many processes.

This process was considered first in a magnetic field. Investigation of pair creation by a photon in a strong magnetic field was started in 1952 independently by Klepikov and Toll [1, 2]. In Klepikov's paper [3], which was based on the solution of the Dirac equation in a constant and homogeneous magnetic field, the probabilities of radiation from an electron and  $e^-e^+$  pair creation by a photon were obtained for the magnetic field of arbitrary strength on the mass shell<sup>1</sup> ( $k^2 = 0$ ,  $k$  is the 4-momentum of photon). In 1971 Adler [4] calculated the photon polarization operator in the mentioned magnetic field using the proper-time technique developed by Schwinger [5] and Batalin and Shabad [6] calculated the photon polarization operator in a constant and homogeneous electromagnetic field for  $k^2 \neq 0$  using the Green function in this field found by Schwinger [5]. In 1975 Strakhovenko and present authors calculated the contribution of a charged-particles loop with  $n$  external photon lines having applied the proper-time method in a constant and homogeneous electromagnetic field [7]. For  $n = 2$  the explicit expressions for the contribution of scalar and spinor particles to the polarization operator of photon are given. Using this polarization operator the integral probability of pair creation by a photon in a magnetic field was analyzed by authors in [8].

The probability of pair creation by a photon in a constant and homogeneous electric field in the quasiclassical approximation was found by Narozhny [9] using the solution of Dirac equation in the Sauter potential [10]. Nikishov [11] obtained the differential cross section of this process using the solution of Dirac equation in a constant and homogeneous electric field.

In the present paper we consider the integral probability of pair creation in an electric field basing on the polarization operator [7]. In Sec.2 the exact expression for the integral probability of pair creation by a photon was obtained for the general case  $k^2 \neq 0$  starting from the polarization operator in an electric field. In Sec.3 the standard quasiclassical approximation (SQA) is outlined for the high-energy photons  $\omega \gg m$  ( $m$  is the electron mass), The corrections to SQA are calculated. These corrections define also the region applicability of SQA. In Sec.4 the new approach is developed for relatively low energies where SQA is not applicable. This approach is based of the method proposed in [8]. The obtained probability is valid in the wide interval of photon energies and is overlapped with SQA. In Sec.5 the case of the very low photon energies  $\omega \ll m$  is analyzed. In particular, in the energy region  $\omega \leq eE/m$  where the previous approach is unapplicable, the low energy approximation is developed. In turn the found results have an overlapping region of applicability with the previous approach. So we have three overlapping approximations which include all photon energies. In conclusion we touch upon the problem connected with the vacuum instability.

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<sup>1</sup>We use the system of units with  $\hbar = c = 1$  and the metric  $ab = a^\mu b_\mu = a^0 b^0 - \mathbf{a} \mathbf{b}$

## 2 Probability of pair creation by a photon: exact theory

Our analysis is based on the expression for the polarization operator obtained in [7], see Eqs.(3.19), (3.33). For pure electric field ( $\mathbf{H}=0$ ) this polarization operation can be written in the diagonal form

$$\Pi^{\mu\nu} = -\sum_i \kappa_i \beta_i^\mu \beta_i^\nu, \quad \beta_i \beta_j = -\delta_{ij}, \quad \beta_i k = 0, \quad \sum_i \beta_i^\mu \beta_i^\nu = \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu}. \quad (1)$$

Here

$$\begin{aligned} \beta_1^\mu &= \frac{k^2 k_\perp^\mu + \mathbf{k}_\perp^2 k^\mu}{k_\perp \sqrt{k^2(\omega^2 - k_3^2)}}, \quad \beta_2^\mu = \frac{F^{\mu\nu} k_\nu}{E k_\perp}, \quad \beta_3^\mu = \frac{F^{*\mu\nu} k_\nu}{E \sqrt{\omega^2 - k_3^2}}, \\ \kappa_1 &= \Omega_1(r - q), \quad \kappa_2 = \kappa_1 + r\Omega_2, \quad \kappa_3 = \kappa_1 - q\Omega_3, \\ r &= \frac{\omega^2 - k_3^2}{4m^2}, \quad q = \frac{k_\perp^2}{4m^2}, \quad r - q = \frac{k^2}{4m^2}, \end{aligned} \quad (2)$$

where the axis 3 directed along the electric field  $\mathbf{E}$ ,  $\mathbf{k}_\perp \mathbf{E} = 0$ ,  $k_\perp = \sqrt{\mathbf{k}_\perp^2}$ ,  $\omega$  is the photon energy,  $F^{\mu\nu}$  is the tensor of electromagnetic field,  $F^{*\mu\nu}$  is the dual tensor of electromagnetic field, and

$$\Omega_i = -\frac{\alpha m^2}{\pi} \int_{-1}^1 dv \int_0^{\infty - i0} f_i(v, x) \exp(i\psi(v, x)) dx. \quad (3)$$

Here

$$\begin{aligned} f_1(v, x) &= \frac{\cosh vx}{\sinh x} - v \frac{\cosh x \sinh vx}{\sinh^2 x}, \quad f_2(v, x) = 2 \frac{\cosh x - \cosh vx}{\sinh^3 x} - f_1(v, x), \\ f_3(v, x) &= (1 - v^2) \coth x - f_1(v, x), \\ \psi(v, x) &= \frac{1}{\nu} \left( 2r \frac{\cosh x - \cosh vx}{\sinh x} - x(1 + q(1 - v^2)) \right), \quad \nu = \frac{E}{E_0}. \end{aligned} \quad (4)$$

Let us note that the integration contour in Eq.(3) is turned slightly down, and in the function  $\Omega_1$  in the integral over  $x$  the subtraction at  $\mu = 0$  is implied.

It should be noted that the probability of pair creation in an electric field (see Eqs.(1)-(4)) can be obtained from the probability of pair creation in a magnetic field (Eqs.(2.1)-(2.5) in [8]) using the formal substitutions  $\mu \rightarrow i\nu$ ,  $x \rightarrow ix$ ,  $q \leftrightarrow -r$ .

The imaginary part of the polarization operator determines the total probability of  $e^- e^+$  pair creation per unit length  $W_i$  by a photon with a given polarization

$$W = \frac{1}{\omega} \text{Im} e^\mu e^{\nu*} \Pi_{\mu\nu}. \quad (5)$$

Using the photon polarization vector  $e_i^\mu = \beta_i^\mu$  we get the expressions for  $W_i$  ( $i = 1, 2, 3$ ) at  $k^2 \neq 0$ :

$$W_i = -\frac{\text{Im}\kappa_i}{\omega} \quad (6)$$

On the mass shell ( $k^2 = 0$ ) one has to put  $r = q$  in Eq.(3). In this case  $W_1 = 0$  and only two photon polarizations  $i = 2$  and  $i = 3$  contribute. In this case the total probability of pair creation averaged over the photon polarizations is

$$W = \frac{W_2 + W_3}{2}. \quad (7)$$

The corresponding analysis in a magnetic field ( $\mathbf{E}=0$ ) was performed in [8]. There are essential differences between pair creation process in magnetic and electric field.

1. The probabilities of pair creation  $W_i(r)$  (for each photon polarization) in a magnetic field contain the factor  $1/\sqrt{g}$  and  $g = 0$  at each threshold when electrons and positrons are created on the Landau levels. Because of this the functions  $W_i(r)$  have saw-tooth form. Rather laborious transformations of the imaginary part of the polarization operator are performed in [8] to obtain the form allowing the direct calculation of the pair creation probabilities. It's more than difficult to use for this purpose directly the expression  $\text{Im}(e^\mu e^{\nu*} \Pi_{\mu\nu})$ . In an electric field there are no levels and  $\text{Im}(e^\mu e^{\nu*} \Pi_{\mu\nu})$  is a smooth function of  $r$  and can be calculated directly using the above equation taking into account that the integration contour in Eq.(3) is turned slightly down. The result for  $\nu = 0.01, 0.001, 0.0001$  is shown in Fig.1. Here and below in all figures the frame  $k_3 = 0$  is used. In general case  $\omega/2m \rightarrow k_\perp/2m = \sqrt{r}$ .
2. In a magnetic field the pair creation is the result of conversion of a photon into pair. The constant and homogeneous magnetic field itself doesn't create a pair. The vacuum is stable. For this reason Eq.(5) gives not only imaginary part of the photon energy but else the overall probability of pair creation. The expression for the polarization operator is valid for field strength  $B > B_0 = m^2/e = 4.41 \cdot 10^{13}G$  (one of possible application of the theory are processes in magnetars with  $B > B_0$ ). In an electric field, generally speaking, pairs can be created by field itself, without a photon presence. The vacuum is unstable. Then Eq.(5) gives the partial probability of pair creation. In the limit  $E \ll E_0 = m^2/e$  ( $\nu \ll 1$ ) the instability of vacuum is negligible. But at  $E \sim E_0$  the vacuum pair creation becomes essential. Even in this situation for  $\omega \gg m$  ( $r \gg 1$ ) the photon create high-energy particles, while the electric field create low-energy particles ( $\varepsilon \sim m$ ) and the pairs are easy distinguishable.
3. In a magnetic field there is the threshold  $\text{Im}(e^\mu e^{\nu*} \Pi_{\mu\nu}) = 0$  for  $r < 1$ . Strictly speaking there is no such requirement in an electric field (see below).

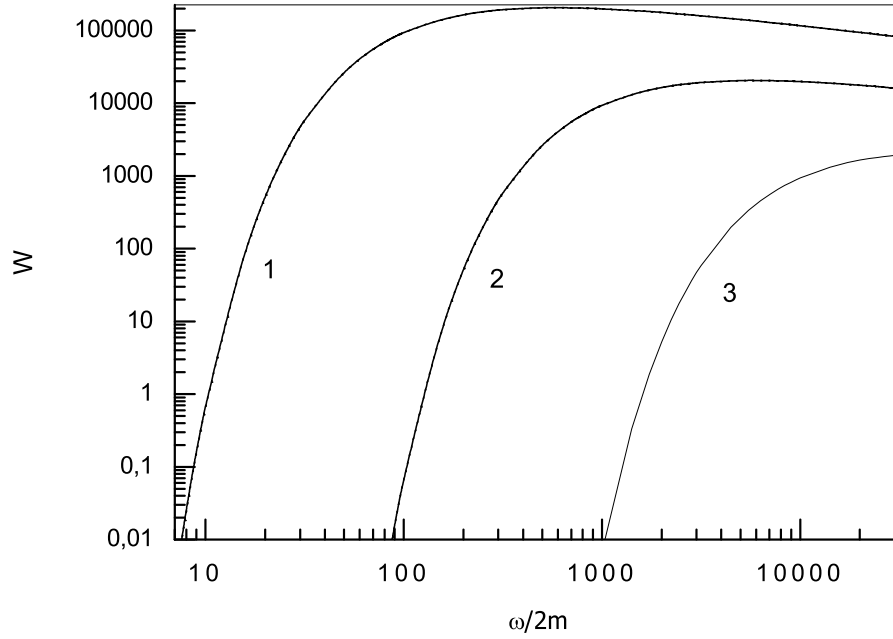


Figure 1: The total probability of pair creation by a photon averaged over the photon polarizations  $W(r)$  (in units  $\text{cm}^{-1}$ ) Eqs.(6), (7) in an electric field for  $\nu = 0.01$  (curve 1),  $\nu = 0.001$  (curve 2),  $\nu = 0.0001$ (curve 3) vs  $\omega/2m$ .

### 3 Probability of pair creation by a photon in quasiclassical approximation

The standard quasiclassical approximation valid for ultrarelativistic created particles ( $r \gg 1$ ) can be derived from Eqs.(3), (4) by expanding the functions  $f_2(v, x)$ ,  $f_3(v, x)$ ,  $\psi(v, x)$  over  $x$  powers. Taking into account the higher powers of  $x$  one gets

$$\begin{aligned}
 f_2(v, x) &= \frac{1-v^2}{12} \left[ -(3+v^2)x + \frac{1}{15}(15-6v^2-v^4)x^3 \right] \\
 f_3(v, x) &= \frac{1-v^2}{6} \left[ (3-v^2)x - \frac{1}{60}(15-2v^2+3v^4)x^3 \right] \\
 \psi(v, x) &= -\frac{r(1-v^2)^2}{12\nu} \left( x^3 - \frac{3-v^2}{30}x^5 \right) - \frac{x}{\nu}.
 \end{aligned} \tag{8}$$

Here the first terms in the brackets give the known probability of the process in the standard quasiclassical approximation, while the second terms are the corrections. Expanding the term with  $x^5$  in  $\exp(i\psi(v, x))$  and making substitution  $x = \nu t$

one finds

$$\begin{aligned}\text{Im}\Omega_i &= i\frac{\alpha m^2 \nu}{2\pi} \int_{-1}^1 dv \int_{-\infty}^{\infty} g_i(v, t) \exp \left[ -i \left( t + \xi \frac{t^3}{3} \right) \right] dx, \\ g_2(v, t) &= \frac{1-v^2}{12} \nu t \left[ -(3+v^2) - i \frac{9-v^4}{90} \xi \nu^2 t^5 + \frac{\nu^2 t^2}{15} (15-6v^2-v^4) \right], \\ g_3(v, t) &= \frac{1-v^2}{6} \nu t \left[ (3-v^2) + i \frac{(3-v^2)^2}{90} \xi \nu^2 t^5 - \frac{\nu^2 t^2}{60} (15-2v^2+3v^4) \right],\end{aligned}\quad (9)$$

where

$$\xi = \frac{(1-v^2)^2 \kappa^2}{16}, \quad \kappa^2 = 4r\nu^2. \quad (10)$$

We will use the known integrals

$$\begin{aligned}\int_{-\infty}^{\infty} \cos \left( t + \xi \frac{t^3}{3} \right) &= \sqrt{3} z K_{1/3}(z), \quad z = \frac{2}{3\sqrt{\xi}} = \frac{8}{3(1-v^2)\kappa}, \\ \int_{-\infty}^{\infty} t \sin \left( t + \xi \frac{t^3}{3} \right) &= \frac{3\sqrt{3}}{2} z^2 K_{2/3}(z).\end{aligned}\quad (11)$$

Conserving the main(first) terms of functions  $g_n(v, t)$  in the integrals Eq.(9) and taking into account Eqs.(10)-(11), we obtain the probabilities of pair creation in standard quasiclassical approximation

$$W_n^{(SQA)} = -\text{Im} \frac{\kappa_n}{\omega} = \frac{\alpha m^2}{3\sqrt{3}\pi\omega} \int_{-1}^1 \frac{s_n}{1-v^2} K_{2/3}(z) dv, \quad s_2 = 3+v^2, \quad s_3 = 2(3-v^2) \quad (12)$$

Here Eq.(12) coincides with the probability obtained in Appendix C [8] for the case of magnetic field. This is because in the quasiclassical approximation the expression for the probability depends on an electromagnetic field via parameter  $\kappa^2 = 4(r\nu^2 + q\mu^2)$  (in the frame where electric and magnetic fields are parallel).

Below we will see that at lower energy the probabilities of pair photoproduction are very different in electric and in magnetic fields.

The probability of pair creation by a photon averaged over photon polarization Eq.(7) calculated in this approximation Eq.(12) coincide with curves found in calculation of exact expressions Eqs.(6)-(7) given in Fig.1. Near maximum of the curves the agreement is better than  $\sim 10^{-5}$ .

The corrections to the standard approximation can be found from the mentioned Appendix C (see Eqs.(C7)-(C12)) by the substitution  $\mu^2 \rightarrow -\nu^2$

$$W_i^{(1)} = -\frac{\alpha m^2 \nu^2}{30\sqrt{3}\pi\omega\kappa} \int_0^1 G_i(v, z) \frac{dv}{1-v^2}, \quad (13)$$

where

$$\begin{aligned} G_2(v, z) &= (36 + 4v^2 - 18z^2)K_{1/3}(z) + (3v^2 - 57)zK_{2/3}(z), \\ G_3(v, z) &= -(34 + 2v^2 + 36z^2)K_{1/3}(z) + (78 - 6v^2)zK_{2/3}(z). \end{aligned} \quad (14)$$

The asymptotic at  $\kappa \gg 1$  are

$$\begin{aligned} W_i^{(1)} &= -\frac{\alpha m^2 \nu^2}{30\sqrt{3}\pi\omega\kappa} w_i, \quad w_2 = 12A\kappa^{1/3} - 90\pi, \\ w_3 &= -11A\kappa^{1/3} + 84\pi, \quad A = 3^{1/3} \frac{2\Gamma^3(1/3)}{5\Gamma(2/3)} = 8.191... \\ W^{(1)} &= \frac{W_1^{(1)} + W_2^{(1)}}{2} = -\frac{\alpha m^2 \nu^2}{60\sqrt{3}\pi\omega\kappa} (A\kappa^{1/3} - 6\pi + ...) \\ \frac{W^{(1)}}{W^{(SQA)}} &= -3^{-7/3} \frac{7}{125} \frac{\Gamma^4(1/3)}{\Gamma^4(2/3)} \frac{\nu^2}{\kappa^{4/3}} \left(1 - \frac{6\pi}{A\kappa^{1/3}} + \dots\right) \end{aligned} \quad (15)$$

At  $\kappa \ll 1$  one has

$$W_2^{(1)} = \frac{\alpha m^2 \nu^2}{\omega \kappa^2} \frac{2\sqrt{2}}{5\sqrt{3}} \exp\left(-\frac{8}{3\kappa}\right), \quad W_3^{(1)} = 2W_2^{(1)}, \quad \frac{W^{(1)}}{W^{(SQA)}} = \frac{32\nu^2}{15\kappa^3}. \quad (16)$$

The curves in Fig.2 characterize the applicability of SQA at energies which are lower than shown in Fig.1. It is seen that for small  $\nu$  the applicability of SQA is broken. For curves 3 and 4, where parameter  $\kappa \ll 1$ , the value  $R(10) - 1$  agrees with the last correction in Eq.(16).

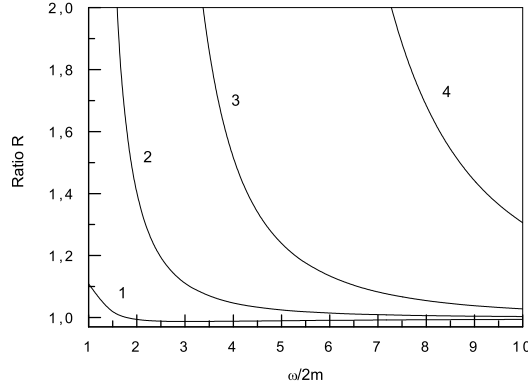


Figure 2: The ratio of total probabilities of pair creation by a photon averaged over the photon polarizations Eq.(7)  $R = W^{(ex)}/W^{(SQA)}$  vs  $\omega/2m$ , where  $W^{(ex)}$  is given by Eqs.(6)-(7),  $W^{(SQA)}$  is given by Eqs.(12),(7). The curve 1 is for  $\nu = 1$ , the curve 2 is for  $\nu = 0.1$ , the curve 3 is for  $\nu = 0.01$ , the curve 4 is for  $\nu = 0.001$ .

## 4 Region of low photon energies

In the field which is weak comparing with the critical field  $E/E_0 = \nu \ll 1$  and at relatively low photon energy ( $r \leq \nu^{-2/3}$ ) the standard SQA [12, 13, 14] is nonapplicable. This follows from the last equality in Eq.(16). In this case, if the condition  $r \gg \nu^2$  is fulfilled, the method of stationary phase at calculation of the imaginary part of the integral over  $x$  in Eq.(3) can be applied. To this end we present the imaginary part of  $\Omega_i$  in the form

$$\text{Im}\Omega_i = i\frac{\alpha m^2}{2\pi} \int_{-1}^1 dv \int_{-\infty}^{\infty} f_i(v, x) \exp[i\psi(v, x)] dx. \quad (17)$$

Granting that the large parameter  $1/\nu$  is the common factor in the phase  $\psi(x)$ , it doesn't contained in the equation  $\psi'(x) = 0$  which defines the stationary phase point  $x_0(r \sim 1) \sim 1$ . In this case the small values of variable  $v$  contribute to the integral over  $v$ , so that one can extend the integration limits to the infinity. So we get

$$\text{Im}\Omega_i \simeq i\frac{\alpha m^2}{2\pi} \int_{-\infty}^{\infty} dv \int_{-\infty}^{\infty} f_i(v = 0, x) \exp\left\{-\frac{i}{\nu} [\varphi(x) + v^2\chi(x)]\right\} dx, \quad (18)$$

where

$$\varphi(x) = 2r \tanh\left(\frac{x}{2}\right) + (r+1)x, \quad \chi(x) = rx \left(-1 + \frac{x}{\sinh x}\right). \quad (19)$$

From the equation  $\varphi'(x) = 0$  we find

$$\tanh\left(\frac{x_0}{2}\right) = -\frac{i}{\sqrt{r}}, \quad \frac{x_0(r)}{2} = -i\frac{a(r)}{2} = -i \arctan \frac{1}{\sqrt{r}}. \quad (20)$$

Substituting these results in the expressions which defines the integral in Eq.(18) we have

$$\begin{aligned} i\varphi(x_0) &= 2 \left( (r+1) \arctan \frac{1}{\sqrt{r}} - \sqrt{r} \right) \equiv b(r), \\ i\varphi''(x_0) &= \frac{r+1}{\sqrt{r}}, \quad i\chi(x_0) = \sqrt{r}a(r)b(r) \\ if_2(v=0, x_0(r)) &= -\frac{r+1}{2r^{3/2}}, \quad if_3(v=0, x_0(r)) = \frac{1}{\sqrt{r}}. \end{aligned} \quad (21)$$

Performing the standard procedure of the method of stationary phase one obtains for probability of pair creation in an electric field by a polarized photon

$$W_2^{(th)} = \frac{\alpha m^2 \nu}{2\omega} \sqrt{\frac{r+1}{ra(r)b(r)}} \exp\left(-\frac{b(r)}{\nu}\right), \quad W_3^{(th)} = \frac{2r}{r+1} \left(1 + \frac{\nu}{4\pi r}\right) W_2^{(th)}, \quad (22)$$



where the term  $\nu/4\pi r$  in  $W_3^{(th)}$  valid at  $r \ll 1$  appears as the contribution of the second term in  $f_1(v, x) (\propto v^2)$  in Eq.(4). These probabilities can be found from the probabilities of pair creation in a magnetic field (see Eqs.(B3)-(B5) (Appendix B in [8] by substitutions  $\mu \rightarrow i\nu$ ,  $r \rightarrow -r$ ,  $\sqrt{-r} = -i\sqrt{r}$ . At this substitution  $l(r) \rightarrow l(-r) = ia(r)$  and  $\beta(r) \rightarrow \beta(-r) = ib(r)$  and from Eq.(B5) in [8] one obtains Eq.(22) (without the correction term  $\nu/4\pi r$ , which was not taken into account in [8]).

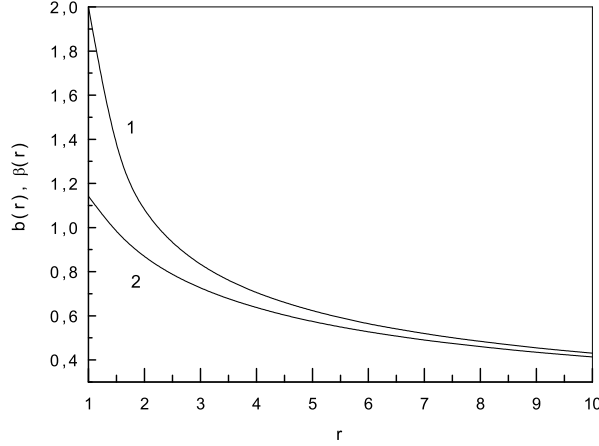


Figure 3: The exponent  $\beta(r)$  in Eq.(B5) (Appendix B in [8]) for pair creation in a magnetic field (curve 1) and the exponent  $b(r)$  in Eq.(22) for pair creation in an electric field (curve 2).

The comparison of the exponent  $b(r)$  with the exponent  $\beta(r) = 2\sqrt{r} - (r - 1)l(r)$ ,  $l(r) = \ln[(\sqrt{r} + 1)/(\sqrt{r} - 1)]$  in the mentioned probability of pair creation in a magnetic field is given in Fig.3. Since the exponent  $b(r)$  enters with very large factor  $1/\nu$  and the exponent  $\beta(r)$  with the factor  $1/\mu$  the probability of pair creation in an electric field is much larger than the probability of pair creation in a magnetic field at  $\nu = \mu \ll 1$ .

In spite of the assumption  $r \sim 1$  made above, Eq.(22) is valid also at  $r \gg 1$  if the condition  $b(r) \gg \nu$  is fulfilled. This can be traced in the derivation of Eq.(22). The first two term of the decomposition of the function  $b(r)$  over power of  $1/r$  are

$$\frac{b(r)}{\nu} \simeq \frac{4}{3\nu\sqrt{r}} - \frac{4}{15\nu r^{3/2}} \quad (23)$$

It follows from this formula that applicability of Eq.(22) is limited by the condition  $r \ll \nu^{-2}$ . If the second term much smaller than unity the exponent with it can be expanded. As a result we have from Eq.(22) at  $\nu^{-2/3} \ll r \ll \nu^{-2}$

$$W_3 = \frac{\alpha m^2 \nu}{2\omega} \sqrt{\frac{3r}{2}} \exp\left(-\frac{4}{3\nu\sqrt{r}}\right) \left(1 + \frac{4}{15\nu r^{3/2}}\right), \quad W_2 = \frac{1}{2}W_3 \quad (24)$$

The main term in the above expression coincides with the probability of pair creation by a photon in the standard quasiclassical theory at  $\kappa = 2\nu\sqrt{r} \ll 1$ . The correction in Eq.(24) determines the lower energy limit of the standard approach applicability ( $\kappa^3 \gg \nu^2$ ) and coincide with Eq.(16). So the overlapping region exists where both the formulated here and the standard approach for high energy are valid.

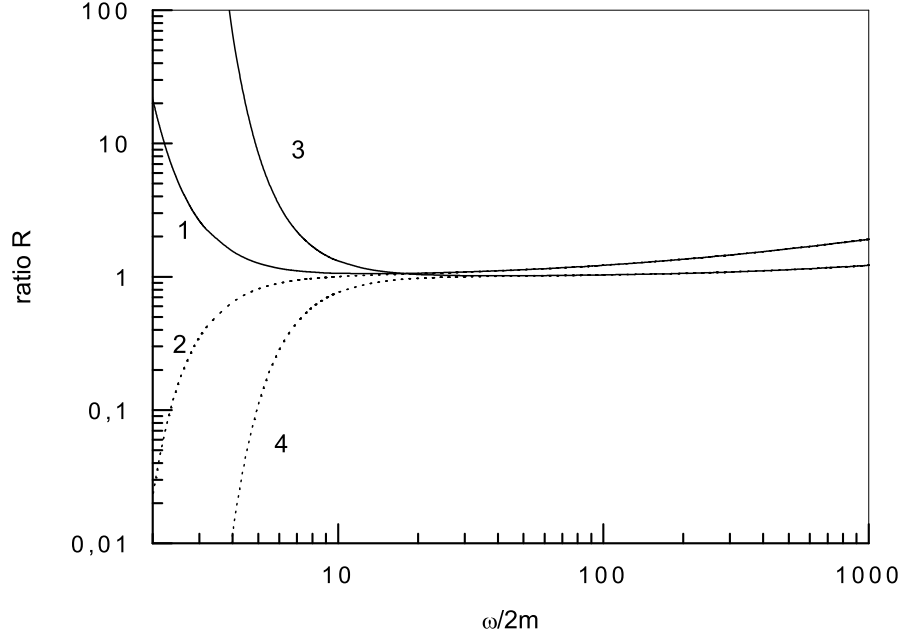


Figure 4: The ratio of the total probabilities of pair creation by a photon averaged over the photon polarizations Eq.(7)  $R = W^{(th)}/W^{(SQA)}$  (see Eqs.(22),(12)) is shown for electric field  $\nu = 0.01$  (curve 1), magnetic field  $\mu = 0.01$  (curve 2) and for electric field  $\nu = 0.001$  (curve 3) and magnetic field  $\mu = 0.001$  (curve 4) vs  $\omega/2m$ .

In Fig.4 the ratio of the probabilities of pair creation by a photon averaged over the photon polarizations Eq.(7)  $R = W^{(th)}/W^{(SQA)}$  (see Eqs.(22),(12)) are given for an electric field (curves 1,3) and a magnetic field (see Eqs.(B3)-(B5) in [8]) (curves 2,4). It is seen that the probabilities in an electric field at low  $r$  values are many order of magnitude higher then in the same magnetic field. The probability in an electric field aims at the standard quasiclassical one with  $r$  increase from above while the probability in a magnetic field aims at the standard quasiclassical value from below. According to Eq.(24), in the interval  $\nu^{-2/3} \ll r \ll \nu^{-2}$  the ratio  $R$  is close to unity for both electric and magnetic fields and for a given  $\nu(\mu)$  the curves have been merged, since in SQA the exponential form of the probabilities

coincide with Eq.(24) (without the correction term which has opposite sign for a magnetic fields). It is valid when  $\kappa \ll 1$ . At  $\kappa = 1$  ( $\sqrt{r} = 1/2\nu$ ) the value  $R \simeq 1.13$  and with further  $r$  increase the value  $R$  smoothly grows. At very high values of  $\kappa$  the probability  $W^{(SQA)} = \alpha m \nu C \kappa^{-1/3}$ ,  $C = 0.37961$  (see [7], Eq.(3.60)). When this equation is valid the value  $R \propto r^{1/6}$ .

At low photon energy ( $\nu^2 \ll r \ll \nu^{2/3}$ ) the probability Eq.(22) has a form

$$W_2 \simeq \frac{\alpha m^2 \nu}{2\pi \omega \sqrt{r}} \left( 1 + \frac{3\sqrt{r}}{\pi} \right) \exp \left( -\frac{\pi}{\nu} (1+r) + \frac{4\sqrt{r}}{\nu} \right), \quad W_3 = \left( 2r + \frac{\nu}{2\pi} \right) W_2. \quad (25)$$

The curves in Fig.5 characterize the applicability of Eq.(22) for different values of parameter  $\nu \ll 1$  in wide interval of  $\omega/2m$ . The region of applicability of the probability  $W^{(th)}$  Eq.(22) is extended with  $\nu$  decrease.

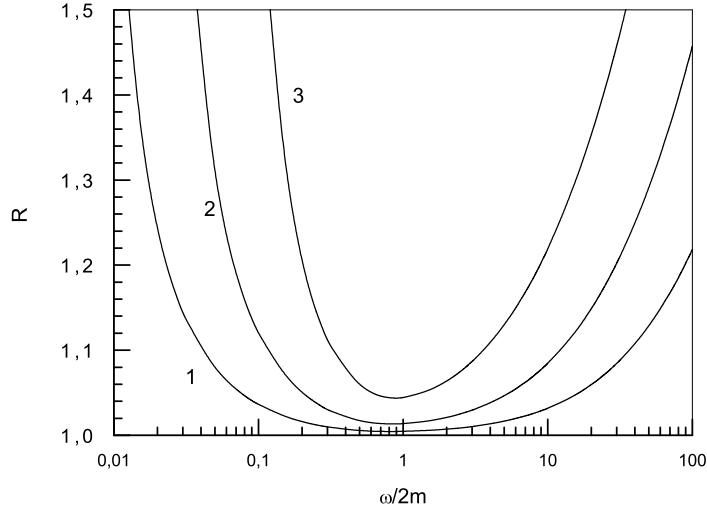


Figure 5: The ratio of the total probabilities of pair creation by a photon averaged over the photon polarizations Eq.(7)  $R = W^{(th)}/W^{(ex)}$  (see Eqs.(22),(6)), for  $\nu = 0.01$  (curve 1), for  $\nu = 0.03$  (curve 2) and for  $\nu = 0.1$  (curve 3) vs  $\omega/2m$ .

## 5 Approximation at very low photon energy

In the case  $r \ll \nu^2/(1+\nu)$  one can expand the exponent in Eq.(17) in powers of the term  $\propto r/\nu$ . Conserving the main term  $\exp(-ix/\nu)$ , independent of variable  $v$ , performing the integration over  $v$  of the functions  $f_i(v, x)$  we find

$$\text{Im}\Omega_i = -i \frac{\alpha m^2}{\pi} \int_{-\infty}^{\infty} \varphi_i(x) \exp(-ix/\nu) dx, \quad (26)$$

where

$$\begin{aligned}\varphi_2(x) &= \frac{2 \cosh x}{\sinh^3 x} - \frac{1}{x \sinh^2 x} - \frac{\coth x}{x^2}, \\ \varphi_3(x) &= \left( \frac{2}{3} - \frac{1}{x^2} \right) \coth x + \frac{1}{x \sinh^2 x}.\end{aligned}\quad (27)$$

The integrals in Eq.(26) can be evaluated closing the integration contour in the lower half-plane and summing the residues in the points  $x = -in\pi$ . Substituting the results into Eq.(2) and then into Eq.(6) we obtain the pair creation probabilities in an electric field in the case  $r \ll \nu^2/(1 + \nu)$

$$\begin{aligned}W_2 &= \frac{2\alpha m^2 r}{\omega} \left[ \frac{1}{\nu^2(e^{\pi/\nu} - 1)} - \frac{1}{\pi\nu} \ln \left( 1 - e^{-\frac{\pi}{\nu}} \right) \right], \\ W_3 &= \frac{2\alpha m^2 r}{\omega} \left[ \frac{2}{3(e^{\pi/\nu} - 1)} + \frac{2}{\pi^2} \text{Li}_2 \left( e^{-\frac{\pi}{\nu}} \right) - \frac{1}{\pi\nu} \ln \left( 1 - e^{-\frac{\pi}{\nu}} \right) \right],\end{aligned}\quad (28)$$

where  $\text{Li}_2(x) = -\int_0^x \frac{\ln(1-t)}{t} dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$  is the Euler dilogarithm.

In the case  $\nu \ll 1$  one has

$$W_2 = \frac{2\alpha m^2 r}{\omega \nu^2} e^{-\frac{\pi}{\nu}} \left( 1 + \frac{\nu}{\pi} \right), \quad W_3 = \frac{2\alpha m^2 r}{\omega \nu^2} e^{-\frac{\pi}{\nu}} \left( \frac{\nu}{\pi} + 2\nu^2 \left( \frac{1}{3} + \frac{1}{\pi^2} \right) \right). \quad (29)$$

At  $\nu \ll 1$  the photon energy region  $r \sim \nu^2$  remains unexplained only. We close the integration contour in the lower half-plane in Eq.(17) in the following way

$$\text{Im}\Omega_i = i \frac{\alpha m^2}{2\pi} \int_{-1}^1 dv \sum_{n=1}^{\infty} \oint f_i(v, x) \exp[i\psi(v, x)] dx, \quad (30)$$

where the path of integration is any simple closed contour around the points  $-i\pi n$ . Expanding the function entering in Eq.(4) over variables  $\xi_n = x + i\pi n$  ( $|\xi_n| \sim \sqrt{r} \sim \nu$ ) and keeping the main terms of decomposition we find

$$\begin{aligned}f_2(v, x) &\simeq -\frac{2}{\xi_n^3} \left[ 1 + (-1)^{n+1} \cos(vn\pi) \right] - f_3(v, x), \quad f_3(v, x) \simeq (-1)^n \frac{iv}{\xi_n^2} \sin(vn\pi); \\ \psi(v, x) &\simeq -\frac{2r}{\xi_n \nu} \left[ 1 + (-1)^{n+1} \cos(vn\pi) \right] - \frac{\xi_n}{\nu} + \frac{i\pi n}{\nu} \left( 1 + r(1 - v^2) \right) \\ &+ \frac{2ir}{\nu} (-1)^n v \sin(vn\pi)\end{aligned}\quad (31)$$

Because of appearance of the factor  $\exp(-\pi n/\nu)$ , in the case  $\nu \ll 1$  the main contribution to the sum in Eq.(30) gives the term  $n = 1$ , then

$$\begin{aligned}f_2(v, x) &= -\frac{4 \cos^2(v\pi/2)}{\xi^3} - f_3(v, x), \quad \psi(v, x) = \frac{4r \cos^2(v\pi/2)}{\xi \nu} - \frac{\xi}{\nu} \\ &+ \frac{i\pi}{\nu} \left( 1 + r(1 - v^2) \right) - \frac{2ir}{\nu} v \sin(v\pi), \quad f_3(v, x) = -\frac{iv}{\xi^2} \sin(v\pi), \quad \xi = \xi_1.\end{aligned}\quad (32)$$

Using the integrals Eq.(7.3.1) and Eq.(7.7.1)(11) in [15] and substituting the result in Eq.(2) and then in Eq.(6) we find

$$W_2 = 2 \frac{\alpha m^2}{\omega} e^{-\frac{\pi(1+r)}{\nu}} \left[ I_1^2 \left( \frac{2\sqrt{r}}{\nu} \right) - \frac{\nu}{\pi} \left( I_0^2 \left( \frac{2\sqrt{r}}{\nu} \right) - 1 \right) + \frac{3\sqrt{r}}{\pi} I_1 \left( \frac{2\sqrt{r}}{\nu} \right) I_0 \left( \frac{2\sqrt{r}}{\nu} \right) \right], \quad W_3 = \frac{\alpha m^2}{\omega} \frac{\nu}{\pi} e^{-\frac{\pi(1+r)}{\nu}} \left( I_0^2 \left( \frac{2\sqrt{r}}{\nu} \right) - 1 \right), \quad (33)$$

where  $I_n(z)$  is the Bessel function of imaginary argument. At calculation of the correction terms  $\propto \nu, \sqrt{r}$  the integration by parts in the integral over  $v$  was performed. The found probability Eq.(33) is applicable for  $r \leq \nu$ . The first term of decomposition of probabilities in Eq.(33) in powers of  $r$  coincide with Eq.(29) if in expression for  $W_3$  the terms  $\propto \nu^2$  are omitted.

For  $r \gg \nu^2$  the asymptotic representation  $I_n(z) \simeq e^z / \sqrt{2\pi z}$  can be used. As a result one obtains the probability Eq.(25). If in Eq.(33) one omits the correction terms  $\propto \nu, \sqrt{r}, r/\nu$ , than one gets the result found in [16] using completely different approach.

The curves in Fig.6 characterize the applicability of Eq.(33) for different values of parameter  $\nu$  in the region of very low energies. It is seen that at  $r > \nu$  Eq.(33) is broken.

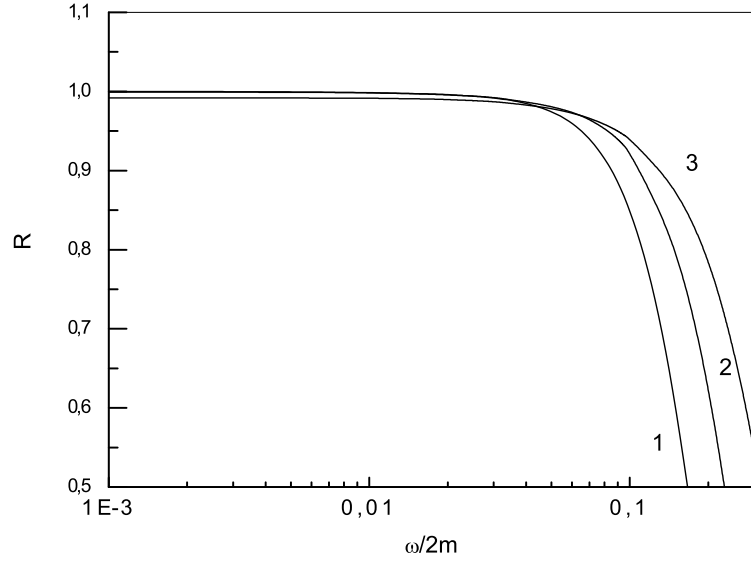


Figure 6: The ratio of the total probabilities of pair creation by a photon averaged over the photon polarizations Eq.(7)  $R = W^{(l)}/W^{(ex)}$ , where  $W^{(l)}$  is given by Eqs.(33),  $W^{(ex)}$  is given by Eq.(6), for  $\nu = 0.01$  (curve 1), for  $\nu = 0.03$  (curve 2) and for  $\nu = 0.1$  (curve 3) vs  $\omega/2m$ .

## 6 Conclusion

We considered the process of pair creation by a photon in an electric field. The probability of the process is calculated using the different approaches. In the case  $\nu \ll 1$  the standard quasiclassical approximation is applicable for the energy parameter  $r \gg \nu^{-2/3}$ . For  $\nu = 0.01$  the averaged over photon polarization probability of pair creation  $W$  Eq.(7) coincides with the standard quasiclassic probability  $W^{(SQA)}$  Eq.(12) within accuracy better than 1% starting from  $r = 160$ . For  $\nu = 0.001$   $W$  coincides with  $W^{(SQA)}$  within accuracy better than 1% starting from  $r = 700$  and for  $\nu = 0.0001$   $W$  coincides with  $W^{(SQA)}$  within accuracy better than 1% starting from  $r = 3200$ . These estimates are in a good agreement with the values of correction term in Eq.(24). Note that always  $W > W^{(SQA)}$ .

Similar situation was observed for the process of pair creation by a photon in a magnetic field [8] where the standard quasiclassical approximation is applicable even in the case  $\mu > 1$  at  $r \gg \mu$ . From Fig.2 it is seen that in an electric field for  $\nu = 1$  the quasiclassical approximation is valid beginning from values  $r$  very close to 1.

The result is universal and the same in electric and magnetic fields if  $\nu = \mu$ . However the corrections to SQA (see Eqs.(14)-(16)) change sign at  $\mu^2 \rightarrow -\nu^2$ .

It should be noted that the consideration based on the polarization operator gives only the total probability of pair creation by a polarized photon (real or virtual). The standard quasiclassical method permits to obtain also the spectral, the angular distributions as well as the polarization of the particles of created pair (see [14]).

For lower values of  $r$  the probability of pair creation in an electric field is much higher than in a magnetic field. Besides the pair creation in an electric field is possible also at  $r < 1$  (although the probability is exponentially small). This phenomenon can be interpreted that an electric field help to draw out the pair from the vacuum. At  $r \ll 1$  ( $\omega \ll m$ ) the inverse situation occurs: a photon helps to electric field to do this work (the factor  $\exp(4\sqrt{r}/\nu)$  Eq.(29)) and for  $r \ll \nu^2$  this aid becomes negligible.

The above analysis is not complete if the probability of direct pair creation by an electric field (vacuum probability) is essential. Then Eq.(6) gives the partial contribution to the probability under consideration and defines the photon lifetime. At  $\nu \ll 1$  ( $E \ll E_0$ ) the vacuum contribution is negligible. But even in this case for very low photon energies in the region  $r \sim \nu^2$  ( $\omega \sim eE/m$ ) the probability Eq.(29) becomes comparable with the vacuum probability.

For lower energies  $r \ll \nu^2$  Eq.(29) defines the probability of photon absorption by the particles of created by an electric field.

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